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Divergent sequences of function groups

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ABSTRACT

In this paper we shall prove a divergence theorem for function groups.

1. Introduction

After the work of W. Thurston in 3-manifold topology and geometry, hyperbolic 3-manifolds and Kleinian group theory became active research areas. His geometrization conjecture spurred up deep researches in many related areas, which is claimed to be settled down by Perelman. Thurston himself proved a famous result, known as the 'Monster Theorem', that atoroidal Haken manifolds admit complete hyperbolic metrics on their interior. When he proved this theorem for manifolds fibering over a circle [30,33], he used his 'Double Limit Theorem' in an essential way. By Ahlfors–Bers theory, there is a covering map from the Teichmüller space of the boundary of the manifold to the space of convex cocompact hyperbolic structures on the interior of the manifold. Then for a sequence of convex cocompact hyperbolic structures, some topological data about the limit point in the corresponding Teichmüller space, guarantees the existence of a limit Kleinian group. Thurston's double limit theorem is a particular case. After his Haken hyperbolization theorem, Thurston conjectured that similar results hold for 3-manifolds with compressible boundary [32]. His conjecture is partially answered by [6,18,29]. Their theorem proves that if the limit projective lamination of the corresponding points in Teichmüller space belongs to the Masur domain and is irrational, then the sequence of representations converges algebraically. See Section 2 for definitions. But recently a generalized version of Thurston's conjecture was given in [17]. They showed that if the sequence of conformal structures at infinity converge to a doubly incompressible projective measured lamination in Thurston's compactification of Teichmüller space, then, up to a subsequence, there is an algebraic limit. For the definition of doubly incompressible measured lamination, see Section 6.

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In this paper we want to deal with the case where there is no convergent subsequence. We prove that if the limit projective lamination lies in some closed set in the complementary region of the Masur domain, then one can construct a divergent sequence in the space of representations.

Let N be a hyperbolic 3-manifold with a compressible boundary S. By Ahlfors–Bers theory there is a covering map $q : \mathcal{T}(\partial N) \to QC(\rho_0)$ where $\mathcal{T}$ denotes Teichmüller space and $QC(\rho_0)$ is a quasi-conformal deformation space of a convex cocompact representation ρ_0. If a sequence $q(m_i, n_i)$, where $m_i \in \mathcal{T}(\partial N \setminus S), n_i \in \mathcal{T}(S)$, has a property that $n_i \to \lambda$ in Thurston compactification of Teichmüller space, then it is said that the sequence converges to the projective lamination λ. The main theorem is:

Theorem. *Let $\mathcal{E} = \{\lambda \in \mathcal{PML}(\partial N) \mid i(\lambda, \partial A) = 0$ for some essential disk or annulus $A\}$. Then for each element λ in $\overline{\mathcal{E}}$, there is a sequence in a convex compact representation space converging to λ, which diverges in $AH(\pi_1(N))$. Here $\overline{\mathcal{E}}$ is the closure of $\mathcal{E}$ in $\mathcal{PML}(\partial N)$ and $AH(\pi_1(N))$ is the space of discrete faithful representations from $\pi_1(N)$ into $PSL(2, \mathbb{C})$ up to conjugacy.*

In this paper, we will give the proof of the main theorem for a compression body for simplicity, but the same proof works for general 3-manifold with at least one compressible boundary.

2. Preliminaries

2.1. Geodesic laminations

A *geodesic lamination* L on a hyperbolic surface S is a nonempty closed subset of S which is a disjoint union of simple geodesics each of which is called a leaf. A *minimal supporting surface* $S(L)$ is a subsurface of S containing L with geodesic boundary, which is minimal up to isotopy. A lamination is called *minimal* if every half leaf is dense. Each lamination can be decomposed as a union of finitely many connected minimal laminations, called minimal components, and finitely many non-compact isolated leaves. The set of geodesic laminations is compact with respect to the topology induced by the Hausdorff distance.

A *measured lamination* is a geodesic lamination with a transverse measure of full support. The support of a measured lamination is a finite union of minimal components. There is a topology on the set $\mathcal{ML}$ of measured laminations which is induced by the intersection form $i : \mathcal{ML} \times \mathcal{ML} \to \mathbb{R}_+$. In this topology, the length function of simple closed curves is continuously extended to $\mathcal{ML}$ [4]. Rescaling the measure provides an action of $\mathbb{R}_+$ on $\mathcal{ML} \setminus \{0\}$. The quotient with the quotient topology is the space of projective measured laminations and is denoted by $\mathcal{PML}$. When S has cusps, $\mathcal{ML}_0(S)$ denotes the set of measured laminations whose supports are compact. See [2] or [9] for more details.

Let S be a compact orientable surface of genus at least 2. Teichmüller space $\mathcal{T}(S)$ is the space of isotopy classes of conformal structures on S. By the Riemann uniformization theorem, it is also the space of isotopy classes of Riemannian metrics on S of constant curvature -1, see [14]. To compactify Teichmüller space, W. Thurston [31] put together objects which are apparently of a very different nature. The topology on the union $\mathcal{T}(S) \cup \mathcal{PML}(S)$ is defined by the property that $m_i \in \mathcal{T}(S)$ tends to $\lambda \in \mathcal{PML}(S)$ if and only if there is $t_i \to 0$ so that for all simple closed curve α on S, $t_i \ell_{m_i}(\alpha)$ tends to $i(\alpha, \lambda)$ where $\ell_{m_i}(\alpha)$ is the infimum of the lengths for m_i of all simple closed curves isotopic to α.

For later uses, we record some lemmata.

Lemma 2.1. *Let S be a hyperbolic surface and $\{t_n^m\}_n$ be a sequence in $\mathcal{T}(S)$ which converges to $\lambda_m \in \mathcal{PML}_0(S)$. If λ_m converges to a λ in $\mathcal{PML}_0(S)$, then there exists $\delta(m) > m$ such that the sequence $\{t_{\delta(m)}^m\}$ converges to $\lambda \in \mathcal{PML}_0(S)$.*

Proof. This follows from a general fact in point set topology, namely the limit of limits is an accumulation point of the union of all the sequences $\{t_n^m\}_n$. □

In [27], the following is introduced. Given a hyperbolic metric σ on S, and fixed constants $0 < \epsilon_1 < \epsilon_0$ which are less than Margulis constant, suppose $\gamma_1, \ldots, \gamma_k$ are simple closed geodesics whose lengths are less than ϵ_1. Then (ϵ_0, ϵ_1)-partial decomposition of σ is the union of components Q, called hyperbolic components, and annular components A_i (which is the complement of hyperbolic components) whose core curve is γ_i with each boundary component having length ϵ_0. One can derive the following easily from [27]. For any annular component A with a core curve γ in (ϵ_0, ϵ_1)-decomposition and a closed geodesic ζ,

$$\left| l_\sigma(\zeta \cap A) - 2\left[\log \frac{\epsilon_0}{l_\sigma(\gamma)} + l_\sigma(\gamma) \frac{Tw_\sigma(\zeta, \gamma)}{2}\right] i(\zeta, \gamma) \right| \leqslant k i(\zeta, \gamma),$$

where $Tw_\sigma(\zeta, \gamma)$ is the absolute value of the twist of ζ around γ defined in [27] (Section 3) (which basically measures the number of twists around γ) and $\log \frac{\epsilon_0}{l_\sigma(\gamma)}$ is approximately half the width of A. Here k depends only on ϵ_0.

For hyperbolic component Q,

$$\left| l_\sigma(\zeta \cap Q) - l_\sigma(\zeta_Q) \right| \leqslant k i(\zeta, \partial Q),$$

where ζ_Q is the geodesic representative of $\zeta \cap Q$ which has the shortest length among the homotopy classes under homotopies that keep end points of arcs on ∂Q. Here k also depends only on ϵ_0. The idea to prove this is that most of twisting ζ around γ takes place in A. Using these one can get, see [11] for example and [12] for similar estimates, that

$$\left|\ell_\sigma(\zeta) - \ sum_j i(\zeta,\gamma_j)\left[2\log\frac{\epsilon_0}{l_\sigma(\gamma_j)} + \ell_\sigma(\gamma_j)Tw_\sigma(\zeta,\gamma_j)\right] - \sum_Q \ell_\sigma(\zeta\cap Q)\right| \leqslant C$$

where C depends only on ϵ_0, (ϵ_0,ϵ_1)-partial decomposition and ζ, independent of σ. Note that once ϵ_0 and (ϵ_0,ϵ_1)-partial decomposition is fixed, $\sum_Q \ell_\sigma(\zeta\cap Q)$ is bounded above, so finally we get

$$\left|\ell_\sigma(\zeta) - \sum_j i(\zeta,\gamma_j)\left[2\log\frac{\epsilon_0}{l_\sigma(\gamma_j)} + \ell_\sigma(\gamma_j)Tw_\sigma(\zeta,\gamma_j)\right]\right| \leqslant C. \tag{1}$$

If c is a simple closed geodesic, then **pinching** (S,σ) along c means that when we represent σ in Fenchel–Nielsen coordinates $(r_i,t_i)_{i=1}^{3g-3}$ with $r_1 = l_\sigma(c)$, we let $r_1 \to 0$ while the other coordinates are fixed. Note that when we pinch along c, $Tw(\cdot,c)$ is constant along the pinching. When C is a system of disjoint simple closed curves, one can define pinching along C similarly.

Lemma 2.2. *Let X be a hyperbolic surface in $\mathcal{T}(S)$. Let c be a simple closed geodesic in X. If X_i is obtained from X by pinching along c so that $l_{X_i}(c) \to 0$, then $X_i \to [c]$ in Thurston compactification. Similarly if $C = r_1c_1 + \cdots + r_kc_k$ is a system of disjoint weighted simple closed curves where r_i is positive real, then there exists a sequence X_i which is obtained from X by pinching along $c_1, c_2, \ldots, c_k$ carefully so that $l_{X_i}(C) = \sum r_i l_{X_i}(c_i) \to 0$ and $X_i \to [C]$.*

Proof. Fix $0 < \epsilon_1 < \epsilon_0$ as above. Let t_i be the reciprocal of the width w_i of the collar A_i of c in X_i each of whose boundary length equals ϵ_0. Since the width goes to infinity, t_i goes to zero. Then for any curve α, it follows from the above equation (1) that

$$\left|l_{X_i}(\alpha) - i(\alpha,c)\left[w_i + l_{X_i}(c)Tw(\alpha,c)\right]\right| \leqslant K$$

for some constant K independent of i. Then for any simple closed curve β in S,

$$t_i l_{X_i}(\beta) \to i(\beta,c).$$

So $X_i \to [c]$ in Thurston's compactification.

For a system of disjoint simple closed curves, $C = r_1c_1 + \cdots + r_kc_k$, pinch X along c_i with different speed to obtain a sequence X_i so that if w_i^j is the width of annuli A_i^j in X_i around c_j each of whose boundary length equals ϵ_0, then $\frac{w_i^j}{w_i^1} \to \frac{r_j}{r_1}$. Set $t_i = \frac{r_1}{w_i^1}$. Then for a simple closed curve α, we get

$$\left|l_{X_i}(\alpha) - \sum i(\alpha,c_j)\left[w_i^j + l_{X_i}(c_j)Tw(\alpha,c_j)\right]\right| \leqslant K.$$

So

$$\lim_{i\to\infty} t_i l_{X_i}(\alpha) = \sum r_j i(\alpha,c_j) = i(\alpha,C).$$

This shows that $X_i \to [C]$ in Thurston's compactification. $\square$

Now we state a fact which is necessary for our main theorem.

Theorem 2.3. *Let λ be a measure lamination on S and C a collection of disjoint simple closed curves so that the support $|\lambda|$ of λ is either disjoint from C or $C \subset |\lambda|$ or $C \cap |\lambda|$ is a union of components of C. Then there is a sequence t_n in $\mathcal{T}(S)$ so that $t_n \to [\lambda]$ and $l_{t_n}(C) \to 0$.*

Proof. (**CASE I**) $\lambda \cap C = \emptyset$.

Take $S(\lambda)$ which is the union of minimal supporting surfaces $S(\lambda^k)$ of components λ^k of λ. Then for each component $S(\lambda^k)$, take a simple closed curve in $S(\lambda^k)$ approximating λ^k in $\mathcal{PML}$. In this way, there exists a sequence of system of weighted simple closed curves λ_i so that $\lambda_i \to \lambda$ in $\mathcal{ML}(S)$. Let $\{\tau_0^i\}$ be a sequence in $\mathcal{T}(S)$ obtained by pinching along C with $\ell_{\tau_0^i}(C) = i^{-1}$. Since $\lambda \cap C = \emptyset$, we can take $\lambda_i \cap C = \emptyset$ for all i. Lemma 2.2 implies that if one pinches along a system of simple closed curves c properly, the sequence converges to $[c] \in \mathcal{PML}$. Let $\{\tau_j^i\}_j$ in $\mathcal{T}(S)$ be a sequence obtained by pinching τ_0^i along λ_i so that $\ell_{\tau_j^i}(\lambda_i) \to 0$. So $\lim_{j\to\infty} \tau_j^i = [\lambda_i]$. Since $\lambda_i \cap C = \emptyset$ for each i, when we pinch τ_0^i along λ_i, we can choose a pants system P which contains λ_i and C so that in Fenchel–Nielsen coordinates with respect to P, only the length of λ_i goes to zero. Then $\ell_{\tau^i_j}(C) = \ell_{\tau^i_0}(C) = i^{-1}$ for all i, j. By Lemma 2.1, there is a sequence $\tau^i_{\delta(i)}$ which converges to $[\lambda]$ and the length of C in $\tau^i_{\delta(i)}$ is i^{-1}. This implies that the length of C in $\tau^i_{\delta(i)}$ tends to 0.

(**CASE II**) $C \subset \lambda$.

As in Case I, we find a sequence of system of weighted simple closed curves λ_i containing C to approximate λ. Pinching along λ_i makes the length of C go to zero since $C \subset \lambda_i$. Then the conclusion follows as before.

(**CASE III**) $C \cap \lambda = c$ is a union of components of C.

Let λ_i be a sequence of system of weighted simple closed curves containing c which converges to λ as in Case I. First construct a sequence τ^i_0 by pinching along C, and then $\{\tau^i_j\}_j$ by pinching τ^i_0 along λ_i so that $\tau^i_j \to [\lambda_i]$. Then as before there exists $\tau^i_{\delta(i)}$ which converges to $[\lambda]$ and $l_{\tau^i_{\delta(i)}}(C) \to 0$ since components of C other than c gets shorter in τ^i_0 and c gets shorter in $\tau^i_{\delta(i)}$. $\square$

2.2. Hyperbolic 3-manifolds

Hyperbolic space $\mathbb{H}^3$ is a complete simply connected Riemannian manifold of constant curvature -1. The Poincaré ball gives a model for hyperbolic space as the unit ball in $\mathbb{R}^3$ with the metric

$$ds^2 = \frac{4dx^2}{(1-r^2)^2}.$$

The boundary of Poincaré ball model is the *sphere at infinity* S^2_∞ for hyperbolic space and the isometries of $\mathbb{H}^3$ prolong to conformal maps on the boundary. The sphere at infinity can be identified with the Riemann sphere $\widehat{\mathbb{C}}$, providing an isomorphism between the orientation preserving isometry group $Isom^+(\mathbb{H}^3)$ and the group of fractional linear transformations $Aut(\mathbb{C}) \cong PSL_2\mathbb{C}$.

A *Kleinian group* Γ is a discrete (torsion free) subgroup of $Isom(\mathbb{H}^3)$. A hyperbolic 3-manifold N is the quotient of hyperbolic 3-space $\mathbb{H}^3$ by a Kleinian group Γ. The *domain of discontinuity* or *discontinuity domain* $\Omega(\Gamma)$ of Γ is the largest Γ-invariant open subset of $\widehat{\mathbb{C}}$ on which Γ acts properly discontinuously. If Γ is not abelian, then $\Omega(\Gamma)$ inherits a hyperbolic metric, called the Poincaré metric, on which Γ acts as a group of isometries. One may then consider $\bar{N} = N \cup \Omega(\Gamma)/\Gamma$ and $\partial_c N = \partial \bar{N} = \Omega(\Gamma)/\Gamma$ to be the *conformal boundary at infinity* of the hyperbolic 3-manifold N. See [1] or [31].

A *compression body* N is a compact oriented 3-manifold which is the boundary connected sum of solid tori and trivial interval bundles over closed surfaces of genus at least 2. A trivial interval bundle over a closed surface is not considered as a compression body. A compression body N is *small* if it is the connected sum along the boundary of either two trivial interval bundles over closed surfaces or an interval bundle over a closed surface and a solid torus. Otherwise it is called *large*. The boundary ∂N of a compression body N has a unique compressible component which is called the *exterior boundary* $\partial_e N$. The inclusion $i_e : \partial_e N \hookrightarrow N$ induces a surjective homomorphism $\pi_1(\partial_e N) \to \pi_1(N)$. The other components except exterior boundary form the interior boundary $\partial_{int} N$.

Given a compression body N, there is a convex cocompact representation $\rho : \pi_1(N) \to PSL_2\mathbb{C}$ such that the interior of N is homeomorphic to $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$ which induces ρ. We say that ρ uniformizes N [24]. The image of ρ is a *function group*, i.e. there is an invariant component of the discontinuity domain Ω_ρ under the action of $\rho(\pi_1(N))$ on $\widehat{\mathbb{C}}$. We can identify the Riemann surface $\Omega_\rho/\rho(\pi_1(N))$ with the boundary ∂N of N. Under this identification the exterior boundary $\partial_e N$ corresponds to a unique invariant component of the discontinuity domain. For more on the topology of compression bodies, see [3] or [26].

2.3. Masur domain

Let N be a compression body with exterior boundary $\partial_e N$ and $\rho : \pi_1(N) \to PSL_2\mathbb{C}$ be a discrete and faithful representation with associated quotient manifold $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$. A *meridian* is a homotopically nontrivial simple closed curve m on the exterior boundary $\partial_e N$ which is compressible in N. A meridian may be seen as an element in the space $\mathcal{PML}(\partial_e N)$ of projective classes of measured lamination on the exterior boundary $\partial_e N$. The set of projective classes of weighted multi-curves of meridians in $\mathcal{PML}(\partial_e N)$ will be denoted by $\mathcal{M}$ and its closure in $\mathcal{PML}(\partial_e N)$ by $\mathcal{M}'$. For a small compression body which is the boundary connected sum of two trivial surface bundles over closed surfaces or the boundary connected sum of a trivial surface bundle over a closed surface and a solid torus, set

$$\mathcal{O} := \{\lambda \in \mathcal{PML}(\partial_e N) \mid i(\lambda, \mu) > 0 \text{ for all } \mu \in \mathcal{PML}(\partial_e N) \text{ such that there is } \nu \in \mathcal{M}' \text{ with } i(\mu, \nu) = 0\}.$$

Otherwise, in a large compression body case, set

$$\mathcal{O} := \{\lambda \in \mathcal{PML}(\partial_e N) \mid i(\lambda, \mu) > 0 \text{ for all } \mu \in \mathcal{M}'\}.$$

The set $\mathcal{O}$ is called the *Masur domain*. We will say that $\lambda \in \mathcal{ML}(\partial_e N)$ is in $\mathcal{O}$ (resp. $\mathcal{M}'$) if its projective class is in $\mathcal{O}$ (resp. $\mathcal{M}'$). Kerckhoff [15] established that the Masur domain has full measure in $\mathcal{PML}(\partial_e N)$.

Let $Mod(N)$ be the group of isotopy classes of homeomorphisms of $\partial_e N$ which extend to homeomorphisms of N and $Mod_0(N)$ be the subgroup of $Mod(N)$ of those elements which are homotopic to the identity in N. $Mod(N)$ acts naturally on $\mathcal{PML}(\partial_e N)$ and acts properly discontinuously on the Masur domain $\mathcal{O}$ (see Masur [23] and Otal [28]). Note that:

Lemma 2.4. *If N is a large compression body, $\mathcal{M}'$ is a unique minimal closed invariant set for the action of $Mod(N)$.*

This lemma due to Otal [28] says that $\mathcal{M}'$ acts like a limit set of $Mod(N)$.

2.4. Deformation spaces of Kleinian groups

Let N be a compact, hyperbolizable 3-manifold and $AH(\pi_1(N))$ denote the space of conjugacy classes of discrete faithful representations of $\pi_1(N)$ into $PSL_2\mathbb{C}$ with *algebraic topology*. The algebraic topology on the set $AH(\pi_1(N))$ is defined by $[\rho_n] \to [\rho]$ if there are representatives of each equivalence class such that $\rho_n(g) \to \rho(g)$ for each $g \in \pi_1(N)$ [33]. Let $QC(\rho_0)$ denote the space of all conjugacy classes of quasi-conformal deformations of ρ_0 where ρ_0 is convex cocompact and uniformizes N. Then a theorem of Marden [24] asserts that every quasi-conformal deformation of ρ_0 is also convex cocompact and uniformizes N.

If N is a compression body with exterior boundary $\partial_e N$ and $\rho_0 : \pi_1(N) \to PSL_2\mathbb{C}$ is a convex cocompact representation, then the space $QC(\rho_0)$ is parameterized by the Teichmüller space $\mathcal{T}(\partial N)$ of the boundary of N. More precisely, there is a covering map

$$\mathcal{T}(\partial N) \to QC(\rho_0)$$

which is called the *Ahlfors–Bers covering*. Teichmüller space $\mathcal{T}(\partial N)$ is the product of the Teichmüller spaces $\mathcal{T}(\partial_e N)$ and $\mathcal{T}(\partial_{int} N)$. The deck transformation group of the Ahlfors–Bers covering is the group $Mod_0(N)$. As homeomorphisms of N preserve the exterior boundary $\partial_e N$, the deck transformation group of the Ahlfors–Bers covering acts on $\mathcal{T}(\partial_e N)$.

We will say that a sequence $\{\rho_i\}$ in $QC(\rho_0)$ *converges into the Masur domain* if it can be parameterized under the Ahlfors–Bers covering by a sequence $(S_i^e, S_i^{int})_i$ in $\mathcal{T}(\partial N) = \mathcal{T}(\partial_e N) \times \mathcal{T}(\partial_{int} N)$ such that $(S_i^e)_i$ converges to a measured lamination $\lambda \in \mathcal{O}$.

2.5. Modulus

We recall that an annulus A is *Euclidean* if it is bounded by two concentric circles. The *core curve* of A is the unique curve invariant by a conformal involution which interchanges the boundary components. The conformal *modulus* of a Euclidean annulus A bounded by circles of radius $r_2 > r_1$ is defined to be

$$mod(A) = \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right).$$

Any (open) topological annulus is conformally equivalent to a Euclidean annulus and we define its modulus to be equal to the modulus of the conformally equivalent Euclidean annulus.

Let γ be a closed geodesic of length L in $\Omega(\Gamma)/\Gamma$. We suppose that γ is compressible and $L \leqslant 1$. Let $\tilde{\gamma}$ be a lift of γ. Canary [8] noticed that $\tilde{\gamma}$ is homotopic to the core curve of a topological annulus $R \subset \Omega(\Gamma)$ with modulus

$$mod(R) \geqslant \frac{\pi}{Le^{\frac{L}{2}}} \geqslant \frac{\pi}{\sqrt{e}L}.$$

A result of Herron, Liu and Minda [13] guarantees that R contains an Euclidean annulus A of modulus close to $mod(R)$.

Theorem 2.5. *(See Corollary 3.5 of [13].) Suppose that R is a topological annulus in $\widehat{\mathbb{C}}$ which separates 0 from ∞. If R has modulus $mod(R) > 0.6$, then R contains a separating Euclidean annulus A, centered at the origin with modulus*

$$mod(A) \geqslant mod(R) - \frac{1}{\pi}\log 2(1+\sqrt{2}) \geqslant mod(R) - 0.502.$$

3. Topology of divergent sequences

Let N be a compression body throughout the rest of paper and $\partial_e N = S_e$ unless otherwise stated. Let's define a domain $\mathcal{C}$ in $\mathcal{PML}(S_e)$ as follows.

$$\mathcal{C} = \bigl\{\lambda \in \mathcal{PML}(S_e) \mid \text{every sequence in } QC(\rho_0) \text{ which converges to } \lambda \text{ has a convergent subsequence in } AH\bigl(\pi_1(N)\bigr)\bigr\}.$$

In other words, the conjecture of Thurston is that Masur domain $\mathcal{O}$ is contained in the convergence domain $\mathcal{C}$. We will prove that $\mathcal{C}$ is open in $\mathcal{PML}(S_e)$. Define $\mathcal{D}$ as

$$\mathcal{D} = \bigl\{\lambda \in \mathcal{PML}(S_e) \mid \text{there exists a divergent sequence } \{\rho_i\} \text{ in } AH\bigl(\pi_1(N)\bigr) \text{ which converges to } \lambda\bigr\}.$$

It follows from the definition of $\mathcal{D}$ that $\mathcal{D}$ is the complement of $\mathcal{C}$ in $\mathcal{PML}(S_e)$. We use the following theorem to find a topological property of $\mathcal{D}$.

Theorem 3.1. *(See Kim [16].) Let $\Re^0(G)$ be the space of irreducible, non-elementary and non-parabolic representations, up to conjugacy, from a finitely presented group G into $\mathrm{Iso}(X)$ where X is a real, complex or quaternionic hyperbolic space. Then there is a smooth embedding f from $\Re^0(G)$ into $\mathbb{R}^k$ and $g_1, \ldots, g_k \in G$ for some k, such that*

$$f(\phi) = \bigl(\ell\bigl(\phi(g_1)\bigr), \ldots, \ell\bigl(\phi(g_k)\bigr)\bigr)$$

for $\phi \in \Re^0(G)$, where $\ell(\phi(g)) = \inf_{x\in X} d_X(x, \phi(g))$ is the translation length of the isometry $\phi(g)$.

In the above theorem, it is important that $\Re^0(G)$ is parameterized by translation lengths of a finite number of elements in G. This enables us to prove the following lemma.

Lemma 3.2. *$\mathcal{D}$ is a closed subset of $\mathcal{PML}(S_e)$.*

Proof. By the above theorem, there exists a finite set $\mathcal{G} = \{g_1, \ldots, g_k\} \subset \pi_1(N)$ such that $AH(\pi_1(N))$ can be parameterized by the above function f. Suppose that $\{\lambda_j\}$ is a sequence in $\mathcal{D}$ which converges to λ in $\mathcal{PML}(S_e)$. For each λ_j, we can find a divergent sequence $\{\rho_i^j\}_i$ in $AH(\pi_1(N))$ which converges to λ_j. Since $\{\rho_i^j\}_i$ diverges in $AH(\pi_1(N))$, the length of one element of $\mathcal{G}$ goes to infinity as i tends to infinity. Let's define the element by $g_{\varphi(j)}$ for each j. Then we obtain a function $\varphi : \mathbb{N} \to \{1, \ldots, k\}$. Since $\mathbb{N}$ is infinite and $\mathbb{N} = \varphi^{-1}(1) \cup \cdots \cup \varphi^{-1}(k)$, there exists an element $g_l \in \mathcal{G}$ such that $\varphi^{-1}(l)$ is an infinite subset of $\mathbb{N}$. Thus we can obtain new series of divergent sequences by relabeling which have the following property:

$$\rho_i^j \to \lambda_j \quad \text{and} \quad \ell\bigl(\rho_i^j(g_l)\bigr) \to \infty \quad \text{as } i \to \infty \quad \text{for each } j.$$

By choosing subsequence of $\{\rho_i^j\}_i$ and relabeling, we may assume that $\ell(\rho_i^j(g_l)) > i$ for all j. By Lemma 2.1, we obtain a sequence $\{\rho^j_{\delta(j)}\}$ which converges to λ in $\mathcal{PML}(S_e)$ and we may assume $\delta(j) > j$. Then $\{\rho^j_{\delta(j)}\}$ diverges in $AH(\pi_1(N))$ because $\ell(\rho^j_{\delta(j)}(g_l)) > \delta(j) > j$ for all j. Therefore $\lambda \in \mathcal{D}$. This means that $\mathcal{D}$ is a closed subset of $\mathcal{PML}(S_e)$. □

4. Meridian case

The length of every meridian tends to infinity in the sequence which converges into Masur domain [18]. By studying a sequence in $AH(\pi_1(N))$ converging to a projective lamination whose support is a meridian m, we will also prove a criterion for divergence of a sequence in $AH(\pi_1(N))$.

Theorem 4.1. *Every sequence in which the length of a meridian goes to zero diverges in $AH(\pi_1(N))$.*

Proof. Let $(S_e)_i$ be the hyperbolic structure on S_e induced by ρ_i. Canary [8] proved that $(S_e)_i$ contains a topological annulus R_i with modulus C/L_i where $C > 0$ is some constant and L_i is the length of a meridian m in $(S_e)_i$. The core curve of R_i is homotopic to m. Suppose that the length L_i of a meridian m in a hyperbolic Riemann surface $(S_e)_i$ goes to zero as i tends to infinity. Since L_i goes to zero, C/L_i goes to infinity. Thus there exists a large K such that $C/L_i > 0.6$ for all $i > K$. Let $\Gamma_i = \rho_i(\pi_1(N))$ and $\widetilde{R}_i$ be a lift of R_i. Then since m is a meridian, one can easily see that R_i and $\widetilde{R}_i$ are biholomorphic by the inclusion map (which is the restriction of a quotient map) $i : \widetilde{R}_i \to (S_e)_i$. Applying Theorem 2.5, $\widetilde{R}_i$ contains a Euclidean annulus $\widetilde{A}_i$ with modulus

$$mod(\widetilde{A}_i) \geqslant mod(\widetilde{R}_i) - 0.502 = \frac{C}{L_i} - 0.502 \quad \text{for all } i > K.$$

This inequality implies that $mod(\widetilde{A}_i)$ goes to infinity as i tends to infinity. Let $CH(\widetilde{A}_i)$ denote the convex hull of $\widetilde{A}_i$. The inclusion map $i : CH(\widetilde{A}_i) \to N_i = \mathbb{H}^3/\rho_i(\pi_1(N))$ is an isometric map onto its image since $i : \widetilde{R}_i \to (S_e)_i$ is an embedding. More precisely, if $x, y \in CH(\widetilde{A}_i)$ are identified in N_i by an element $\gamma \in \pi_1(N)$, by the definition of a convex hull, there are two totally geodesic hyperplanes H_1, H_2 contained in $CH(\widetilde{A}_i)$ and containing x, y respectively so that $\gamma(\partial H_1 \cap \widetilde{A}_i) \cap (\partial H_2 \cap \widetilde{A}_i) \neq \emptyset$. This would imply that $i : \widetilde{R}_i \to (S_e)_i$ is not an embedding.

Suppose γ is an element of $\pi_1(N)$ with $i(\rho_i(\gamma)^*, m^*) \neq 0$ where $\rho_i(\gamma)^*$ and m^* are the geodesic representatives of $\rho_i(\gamma)$ and m in $(S_e)_i$ respectively. Note that the length of shortest geodesic arc in N_i passing through $CH(\widetilde{A}_i)$ is at least $2\pi\, mod(\widetilde{A}_i)$. Thus we obtain the following inequality.

$$\ell_{N_i}(\gamma) > \ell_{N_i}\bigl(\gamma \cap i\bigl(CH(\widetilde{A}_i)\bigr)\bigr) \geqslant 2\pi\, mod(\widetilde{A}_i) > 2\pi\left(\frac{C}{L_i} - 0.502\right).$$

Therefore $\{\rho_i\}$ is a divergent sequence in $AH(\pi_1(N))$ because $\ell_{N_i}(\gamma)$ goes to infinity. □

Above theorem can be proved by using Theorem 5.4 equally. Here we can argue that forming a long neck in the convex core forces the sequence diverge.

5. Essential annulus case

We start with an illuminating example. Two Kleinian groups Γ^* and Γ, with invariant components Δ^* and Δ respectively, are called *weakly similar* if there is an orientation preserving homeomorphism $\varphi : \Delta^* \to \Delta$, where $\varphi \circ g \circ \varphi^{-1}$ defines an isomorphism from Γ^* onto Γ. In this case, φ is called a *weak similarity*. If the isomorphism $g \to \varphi \circ g \circ \varphi^{-1}$ is type preserving, then φ is a *similarity*, and Γ^* and Γ are called *similar*; if in addition, φ is conformal, then Γ^* and Γ are called *conformally similar*. Recall that a *function group* is a finitely generated Kleinian group with an invariant component in the domain of discontinuity. If N is a compression body, $\pi_1(N)$ can be written as $\Gamma_1 * \cdots * \Gamma_k$ where Γ_i is isomorphic to the fundamental group of a factor manifold in N which is either a trivial interval bundle over a closed surface or a solid torus. Each Γ_i is called a *structure subgroup*. A function group in which every structure subgroup is either Fuchsian or elementary is called a *Koebe group*. Every structure subgroup of a Koebe group is geometrically finite. This means that a Koebe group is geometrically finite. See [20] for details.

Theorem 5.1. *(See Maskit [21].) Let Γ be a function group. Then there is a unique Koebe group Γ^*, and there is a unique (up to elements of $PSL_2(\mathbb{C})$) conformal similarity between Γ and Γ^*.*

Example 5.2. We can find a divergent sequence in the deformation space $AH(\pi_1(N))$, converging to a projective measured lamination disjoint from the boundary of some essential annulus as follows, using Koebe groups. Let $N = \mathbb{H}^3/\Gamma$ be a convex cocompact compression body, and ρ_0 a representation with image Γ. Let S be a component of the incompressible conformal boundary of N and γ be a simple closed curve in S_e homotopic to γ' in S. From the 3-manifold theory, there is an embedded essential annulus A which bounds γ and γ'. Let λ be a simple closed curve in S_e disjoint from γ. Fix a hyperbolic metric σ on S_e. Now let τ_0^j be the hyperbolic surface obtained from (S_e, σ) by pinching along γ so that $\ell_{\tau_0^j}(\gamma)$ tends to zero.

Now set $\{\tau_i^j\}_i$ to be a sequence obtained from τ_0^j by pinching along λ so that $\ell_{\tau_i^j}(\lambda) \to 0$. Then by Lemma 2.2, $\lim_{i\to\infty} \tau_i^j = [\lambda]$. One can find $\delta(j) > j$ so that $\tau_j = \tau_{\delta(j)}^j \to [\lambda]$ by Lemma 2.1. The lengths of γ and λ in $\{\tau_j\}$ tends to zero since γ and λ are disjoint. Using Ahlfors–Bers covering map $q : \mathcal{T}(\partial N) \to QC(\rho_0)$, one can find a function group Γ_j' whose conformal structure on S_e equals τ_j. Then Theorem 5.1 implies that there is a unique Koebe group Γ_j which is conformally similar to Γ_j', having the conformal structure τ_j on the exterior boundary S_e. One can notice that Koebe group Γ_j is in $QC(\rho_0)$. Let $N_j = \mathbb{H}^3/\Gamma_j$. Maskit [22] proved that if Λ is any Kleinian group,

$$\ell_M(c^*) \leqslant \pi \ell_Y(c) e^{\ell_Y(c)/2},$$

where $M = \mathbb{H}^3/\Lambda$ and c is any closed curve on $Y = \Omega(\Lambda)/\Lambda$. Applying this to the Koebe group Γ_j, we obtain that $\ell_{N_j}(\gamma^*)$ goes to zero as j tends to infinity because $\ell_{\tau_j}(\gamma)$ goes to zero. If G_j is the Fuchsian group corresponding to the structure subgroup of Γ_j which is isomorphic to $\pi_1(S)$, then there is a covering $p : N_j^S = \mathbb{H}^3/G_j \to N_j$. Let the conformal boundary of the Fuchsian group G_j be $X_j \in \mathcal{T}(S)$. Since p is a Riemannian covering map and G_j is a Fuchsian group, we get $\ell_{X_j}(c) = \ell_{N_j^S}(c^*) = \ell_{N_j}(c^*)$ for all simple closed curves c in S. The geodesic representatives of γ and γ' in N_j coincide and so the length of γ' in the sequence $\{X_j\}$ also goes to zero. Let α be a simple closed curve intersecting γ' in S transversely. Then the length of α in X_j goes to infinity because $\ell_{X_j}(\gamma')$ goes to zero. Thus $\ell_{N_j}(\alpha^*) = \ell_{X_j}(\alpha)$ goes to infinity as j tends to infinity. Therefore we can conclude that the sequence Γ_j diverges in $AH(\pi_1(N))$.

Remark 5.3. In the above example, we see that as the length of ∂A gets smaller and smaller, the length of some curve intersecting A transversely becomes longer and longer for a specific essential annulus A. We will generalize this for arbitrary essential annulus A.

5.1. Compression body which does not contain a solid torus

Theorem 5.4. *(See Canary [8].) Suppose that N is a hyperbolic 3-manifold and $r : \partial_c N \to \partial C(N)$ is the nearest point retraction from its conformal boundary to the boundary of its convex core. If α is a closed curve in the conformal boundary of length L, then*

$$\ell_{\partial C(N)}\bigl(r(\alpha)^*\bigr) < 45 L e^{\frac{L}{2}}$$

where $\ell_{\partial C(N)}(r(\alpha)^*)$ denotes the length of the closed geodesic in the intrinsic metric on $\partial C(N)$ in the homotopy class of $r(\alpha)$.

Lemma 5.5. *Let N be a compression body and A be an essential annulus in N. Suppose that no component of $N - A$ is a solid torus. Then there exists $\gamma \in \pi_1(N)$ such that any curve homotopic to γ must intersect A.*

Proof. If A does not separate N, there exists a simple loop γ such that γ intersects A at only one point. Then one can easily see that this curve γ satisfies the property which is required in this lemma by the 3-manifold theory. Now suppose that A

separates N into two pieces, V and W. Since neither V nor W is a solid torus, there are primitive elements $\gamma_1 \in \pi_1(V)$ and $\gamma_2 \in \pi_1(W)$ both of which are not homotopic to any element in $\pi_1 A$. Take $\gamma = \gamma_1 * \gamma_2$. If there is a curve γ' homotopic to γ which does not intersect A, then γ' is contained in either V or W. This implies γ is an element of either $\pi_1(V)$ or $\pi_1(W)$ because γ is homotopic to γ'. It is contradiction to the construction of γ. Thus we conclude that any curve homotopic to γ must intersect A. □

Theorem 5.6. *Let A be an essential annulus in N. Suppose that no component of $N - A$ is a solid torus. Then every sequence of convex cocompact representations in which the length of ∂A in the conformal boundary tends to zero diverges in $AH(\pi_1(N))$.*

Proof. Let $\partial A = \alpha_1 \cup \alpha_2$ and $\{\rho_i\}$ be a sequence in $QC(\rho_0)$ and $N_i = \mathbb{H}^3/\rho_i(\pi_1(N))$. Suppose that the length of ∂A in the conformal exterior boundary $(S_e)_i$ of N_i tends to zero as i goes to infinity. Assume the sequence $\{\rho_i\}$ converges in $AH(\pi_1(N))$. Let γ be the simple closed curve which is given in Lemma 5.5 for A. The convex core $C(N)$ of a hyperbolic 3-manifold is the smallest convex submanifold such that the inclusion of $C(N)$ into N is a homotopy equivalence. The convex core $C(N)$ of a convex cocompact manifold N is homeomorphic to $\bar{N}$ and $N - C(N)$ is homeomorphic to $\partial_c N \times (1, \infty)$. Then the intrinsic metric on the boundary $\partial C(N)$ of the convex core is hyperbolic and the nearest point retraction $r : \partial_c N \to \partial C(N)$ gives a proper homotopy equivalence. Thus there is an essential annulus A_i contained in $C(N_i)$ and $\partial A_i = (\alpha_1)_i \cup (\alpha_2)_i$ is contained in $\partial C(N_i)$ where $(\alpha_k)_i$ is the simple closed curve in $\partial C(N_i)$ corresponding to the geodesic representative of α_k in the hyperbolic surface $\partial C(N_i)$ for $k = 1, 2$. Moreover we may assume that A_i is a simplicial ruled annulus by straightening the homotopy of $(\alpha_1)_i$ to $(\alpha_2)_i$. By the property of γ, the geodesic representative γ_i^* of γ in N_i must intersect A_i. Now we will show that both γ_i^* and A_i lie in the same component of a thin part of N_i, which will imply that γ and $\pi_1 A$ commute in $\pi_1 N$. So $\pi_1 N$ will contain $\mathbb{Z} \oplus \mathbb{Z}$. From this we will derive a contradiction.

One can estimate the area of the simplicial ruled annulus as follows.

Lemma 5.7. *(See Canary [7].) If A is a simplicial ruled annulus with boundary $C_1 \cup C_2$ in a manifold N with pinched negative curvature $(\leqslant -1)$, then*

$$\mathit{area}(A) \leqslant \ell(C_1) + \ell(C_2).$$

Moreover,

$$\mathit{area}\big\{x \in A \mid d(x, C_1) \geqslant D\big\} \leqslant \ell(C_2) + e^{-D}\ell(C_1).$$

Since the sequence $\{\rho_i\}$ converges in $AH(\pi_1(N))$ and γ_i^* is the geodesic representative of $\gamma \in \pi_1(N)$ in N_i, there is a constant $B > 0$ such that $\ell_{N_i}(\gamma_i^*) \leqslant B$ for all i. From Theorem 5.4,

$$\ell_{\partial C(N)}\big(r(\alpha_k)^*\big) < 45 L_k^i e^{\frac{L_k^i}{2}} \quad \text{for } k = 1, 2 \text{ where } L_k^i = \ell_{(S_e)_i}(\alpha_k).$$

The right-hand side tends to zero because L_k^i tends to zero as i goes to infinity and clearly $r(\alpha_k)^* = (\alpha_k)_i$. Thus

$$\ell_{\partial C(N_i)}\big((\alpha_k)_i\big) \to 0 \quad \text{as } i \to \infty \text{ for } k = 1, 2.$$

It immediately follows that area(A_i) tends to zero by Lemma 5.7. This means that $\sup_{x \in A_i} inj_{N_i}(x)$ goes to zero and so there is a sequence $\{\epsilon_i\}$ converging to 0 such that $\sup_{x \in A_i} inj_{N_i}(x) < \epsilon_i$ for all i. Then it is more or less obvious that γ_i^* and A_i lie in the same component of a thin part of N_i. Since $\sup_{x \in A_i} inj_{N_i}(x) < \epsilon_i$ is small for large i, and since the radius of the Margulis tube of the core of A_i is going to infinity, A_i lies deep inside this Margulis tube. Furthermore since γ_i^* must intersect A_i and $l_{N_i}(\gamma_i^*)$ is uniformly bounded, it is geometrically obvious that γ_i^* also lie in this Margulis tube. We refer the readers to [5].

This implies that both A_j and γ_j^* lie in the same component of μ-thin part of N_j, which violates the fact that there is no rank 2-cusp in N_j. Note that N_j is convex cocompact. Thus the sequence $\{\rho_i\}$ does not converge in $AH(\pi_1(N))$. □

5.2. Compression body which contains a solid torus

A *pared manifold* (M, P) consists of a compact, irreducible, oriented 3-manifold M and a collection P of disjoint incompressible annuli and tori in ∂M such that

1. If A is an abelian subgroup of $\pi_1(M)$ which is not cyclic, then A is conjugate into the fundamental group of a component of P, and
2. Every map $\phi : (S^1 \times I, S^1 \times \partial I) \to (M, P)$ that is injective on the fundamental groups, is homotopic, as a map of pairs, into P.

Let N_ϵ^0 be obtained from N by removing the unbounded components of its ϵ-thin part. We say that $(\mathcal{M}, \mathcal{P})$ is a *relative compact core* for N_ϵ^0 if $\mathcal{M}$ is a compact core for N_ϵ^0 and $\mathcal{M}$ intersects each component of ∂N_ϵ^0 in a connected submanifold which is a compact core for that component. For the existence of such a compact core, see [25].

Theorem 5.8. *Let A be an essential annulus in N. Let $i(\lambda, \partial A) = 0$ where a component of $N - A$ is a solid torus. Then $\lambda \in \mathcal{D}$.*

Proof. Suppose λ is a simple closed curve. Let V be a solid torus which is a component of $N - A$. Then $\partial V - A$ is an annulus on ∂N because ∂V is a torus. Thus A is not homotopic to $\partial V - A$ by the definition of an essential annulus. Let γ be a component of ∂A. If $\lambda \cap (\partial V - A) \neq \emptyset$, then λ is homotopic to γ because $i(\lambda, \gamma) = 0$. Thus λ is homotopic to γ or $\lambda \cap (\partial V - A) = \emptyset$. □

Lemma 5.9. *Let B be an annulus in N which cuts off a solid torus V. Let $A = \partial V - B$ be an annulus in ∂N with core curve γ which is nontrivial in $\pi_1 V$. Then A is not homotopic to $B = \partial V - A$ if and only if $i(\gamma, m) \geqslant 2$ where m is a meridian in the solid torus. Moreover, if $i(\gamma, m) \geqslant 2$, then $B = \partial V - A$ is an essential annulus in N.*

Proof. Suppose that $(A, \partial A)$ is homotopic to $(\partial V - A, \partial(\partial V - A))$ as a map of pairs in N. Then let $H : A \times I \to V$ be a homotopy from A to $\partial V - A$ where $I = [0, 1]$ such that $H|_{\partial A \times I} = Id|_{\partial A \times I}$ and $H(A \times 0) = A$ and $H(A \times 1) = \partial V - A$. Since $A \times I$ is a solid torus and H is a proper degree 1 map, the claim of the first statement follows.

Now suppose that $i(\gamma, m) \geqslant 2$. We will prove that $\partial V - A$ is an essential annulus in N. If $(\partial V - A, \partial A)$ is homotopic to an annulus $(A', \partial A)$ in ∂N as a map of pairs, then $A' \cap A = \partial A$ because A' cannot be A by the above argument, it follows that ∂N contains a torus component $A \cup A'$. It is impossible since N is a compression body. Thus $\partial V - A$ is not homotopic to any annulus in ∂N. This means that $\partial V - A$ is an essential annulus in N. □

Continued the proof of Theorem 5.8. By the above lemma, $i(m, \gamma) \geqslant 2$. Let P be a pants-decomposition of $\partial_e N$ containing γ and λ so that any two different simple closed curves in P are not freely homotopic in N. Due to Theorem 2.3, there is a sequence $\{\tau_n\}$ in $\mathcal{T}(S_e)$ satisfying the following properties:

(1) $\{\tau_n\}$ converges to $[\lambda]$ in $\mathcal{PML}(S_e)$,
(2) $\ell_{\tau_n}(P) \to 0$ as $i \to \infty$.

For fixed hyperbolic structures on incompressible boundaries and τ_n on S_e, we get a sequence $\{\rho_n\}$ in $QC(\rho_0)$ by the Ahlfors–Bers covering. Suppose that $\{\rho_n\}$ converges to ρ in $AH(\pi_1(N))$. Let $N_\rho = \mathbb{H}^3/\rho(\pi_1(N))$. Then ρ has accidental parabolics corresponding to P.

Lemma 5.10. *Suppose that N is a hyperbolic 3-manifold with finitely generated fundamental group, $\epsilon < \mu$, and $(\mathcal{M}, \mathcal{P})$ is a relative compact core for N^0_ϵ where μ is the Margulius constant. If a component F of $\partial\mathcal{M} - \mathcal{P}$ is a thrice punctured sphere, then the associated end of N^0_ϵ is geometrically finite.*

See [10] for example. Let $(\mathcal{M}_\rho, \mathcal{P}_\rho)$ be a relative compact core for $(N_\rho)^0_\epsilon$. Since $\mathcal{P}_\rho$ consists of annulus neighborhoods of P by the geometric definition of algebraic convergence, every component of $\partial\mathcal{M}_\rho - \mathcal{P}_\rho$ is thrice punctured sphere except irreducible components corresponding to $\partial_{int} N$. It means that every end of $(N_\rho)^0_\epsilon$ is geometrically finite. Thus N_ρ is geometrically finite. Now Lemma 5.9 implies that $\pi_1(N)$ is an amalgamation of a group $\pi_1(N \setminus V)$ and $\mathbb{Z}$ where we are amalgamating over the nth power of the generator of $\mathbb{Z}$ with $n \geqslant 2$. As the solid torus will be part of the characteristic submanifold, the generator of $\mathbb{Z}$ will never be a peripheral element of $\pi_1(N')$ for any hyperbolic 3-manifold N' homotopy equivalent to N. However under ρ, this generator is parabolic and it must be peripheral. It is a contradiction. Thus we can conclude that $\{\rho_n\}$ diverges in $AH(\pi_1(N))$ and so $\lambda \in \mathcal{D}$.

Now suppose that λ is a general measured lamination with $i(\lambda, \gamma) = 0$. Take $S(\lambda)$ which is the union of minimal supporting surfaces $S(\lambda^k)$ of components λ^k of λ. Then for each component $S(\lambda^k)$, take a simple closed curve in $S(\lambda^k)$ approximating λ^k in $\mathcal{PML}$. In this way, there exists a sequence of system of simple closed curves λ_i so that $\lambda_i \to \lambda$ in $\mathcal{PML}(S_e)$ with $i(\lambda_i, \gamma) = 0$. Since $\lambda_i \in \mathcal{D}$ and $\mathcal{D}$ is closed, λ is also in $\mathcal{D}$. □

6. Main theorem

Theorem 6.1. *Let $\mathcal{E} = \{[\lambda] \in \mathcal{PML}(\partial_e N) \mid i(\lambda, \partial A) = 0$ for some essential disk or annulus $A\}$. Then for each element $[\lambda]$ in $\overline{\mathcal{E}}$, there is a sequence converging to $[\lambda]$, which diverges in $AH(\pi_1(N))$.*

Proof. If we prove that $\mathcal{E} \subset \mathcal{D}$, then since $\mathcal{D}$ is closed, the closure of $\mathcal{E}$ is contained in $\mathcal{D}$. Thus we only prove that $[\lambda]$ is contained in $\mathcal{D}$ for $[\lambda] \in \mathcal{E}$. □

Let $i(\lambda, \partial A) = 0$ for an essential disk or annulus A. By Theorem 5.8, we assume that no component of $N - A$ is a solid torus when A is an essential annulus. Then it implies that either $\partial A \subset \lambda$ or $\lambda \cap \partial A = \emptyset$ or $\partial A \cap \lambda$ is a boundary component of an essential annulus. In all cases one can find a sequence in $\mathcal{T}(S_e)(S_e = \partial_e N)$ which converges to $[\lambda]$ and the length of ∂A goes to zero by Theorem 2.3, which implies $[\lambda] \in \mathcal{D}$ using Ahlfors–Bers covering map and Theorem 4.1, Theorem 5.6.

Let $\mathcal{D}(N) = \{\lambda \in \mathcal{ML}(\partial N) \mid$ there exists a $\eta > 0$ such that for any essential disk or annulus A, $i(\partial A, \lambda) > \eta\}$. In [17], an element in $\mathcal{D}$ is called doubly incompressible. Lecuire proved the interesting theorem for $\mathcal{D}(N)$ as follows [19].

Theorem 6.2 *(Lecuire). If M is not a genus two handlebody, then $\mathcal{D}(N)$ is the domain of discontinuity for the action of $Mod(N)$ on $\mathcal{ML}(\partial N)$.*

Note that $\mathcal{E}$ is invariant under $Mod_0(N)$, so its closure is invariant also. So it must contain $\mathcal{M}'$ by Lemma 2.4. We have two important $Mod_0(N)$-invariant closed subsets in $\mathcal{PML}(S_e)$. One of them is the complement of the Masur domain and the other is the complement of $\mathcal{D}_p(N)$ where $\mathcal{D}_p(N)$ is the projection of $\mathcal{D}(N)$ on $\mathcal{PML}(\partial N)$. But in general, Masur domain and doubly incompressible domain do not coincide. Here is an example.

Lemma 6.3. $\mathcal{D}_p(N)^c \subsetneqq \mathcal{O}^c$.

Proof. It suffices to show that $\mathcal{O} \subsetneqq \mathcal{D}_p(N)$. Let S be a compact surface with one boundary component with higher genus. Then $N = S \times I$ is a handle body of the genus twice that of S. Set c be the core curve of $\partial S \times I$. Take two sets of systems of disjoint simple closed curves, X and Y so that the curves in X lie on $S \times 1$ and the ones in Y on $S \times 0$, so that $X \cup Y$ binds S, i.e., $S \setminus X \cup Y$ is a union of disks and the annulus around the boundary of S. Then we claim that $X \cup Y$ is in $\mathcal{D}(N)$. For any essential disk A, since S is incompressible in N, ∂A should intersect c transversely. This means that ∂A should intersect some curve in $X \cup Y$ transversely. So intersection number between a meridian and $X \cup Y$ is at least 1. Similar thing holds for any essential annulus. So $X \cup Y$ is in $\mathcal{D}(N)$.

Now we claim that $X \cup Y$ is not in $\mathcal{O}$. Take any meridian m. Since S is incompressible, m should intersect c transversely. Then wrapping m around c with appropriate weights will give an element in $\mathcal{M}'$ whose support is c. Then $X \cup Y$ does not intersect c, so it is not in $\mathcal{O}$.

Note here that if we take S to be a torus with n boundary components, then the handle body will have genus $n + 1$, so we can construct an example with arbitrary genus. $\square$

There is a relation between these closed sets as follows.

$$\mathcal{M}' \subsetneqq \overline{\mathcal{E}} \subset \mathcal{D}_p(N)^c \subsetneqq \mathcal{O}^c.$$

The domain $\mathcal{D}$ is also a closed subset containing $\overline{\mathcal{E}}$ and by the result of [17], $\mathcal{D}$ is contained in $\mathcal{D}_p(N)^c$.

Open question. We conjecture that $\mathcal{D} = \overline{\mathcal{E}} = \mathcal{D}_p(N)^c$.

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